BEHR: Bayesian Estimation of Hardness Ratios

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Abstract

- We present a generalized and statistically coherent scheme of computing hardness ratios and their associated error bars.
- In this scheme, we model the observed photons as independent Poisson variables and exploit sophisticated Bayesian approaches (e.g., Gibbs sampling) to calculate hardness ratios, accounting for local background contamination and effective area variations.
- We apply this scheme to the simple counts ratio $[S/H]$ as well as its variants, X-ray colors $[\log_{10}(S/H)]$ and fractional difference hardness ratios $[(H-S)/(H+S)]$.
- We also perform simulations to compare the new Bayesian methods with the classical method, thereby illustrating that (a) the former provides more accurate estimates of the uncertainties, (b) the mode of the posterior probability distribution function (pdf) represents each hardness ratio the best, and (c) the pdfs of the colors are the best behaved in the low counts limit.
- Finally, we apply this method to identify candidate qLMXB's in the globular cluster Terzan 5.

The Classical Method

A Hardness Ratio:

- Given observed counts in the soft band (S) and the hard band (H), a hardness ratio can be computed as a summary of a spectrum:
  1. Simple counts ratio, $\mathrm{R} = \dfrac{S}{H}$,
  2. X-ray color, $\mathrm{C} = \log_{10}\dfrac{S}{H}$, and
  3. Fractional difference hardness ratio, $\mathrm{HR} = \dfrac{H-S}{H+S}$.
- In the presence of background where B_S and B_H are collected in an area of c times the source region, the above is generalized to
  1. $\mathrm{R} = \dfrac{S - B_S/c}{H - B_H/c}$,
  2. $\mathrm{C} = \log_{10}\left(\dfrac{S - B_S/c}{H - B_H/c}\right)$, and
  3. $\mathrm{HR} = \dfrac{(H - B_H/c) - (S - B_S/c)}{(H - B_H/c) + (S - B_S/c)}$,

  and their errors are computed **under Gaussian assumptions**:
  1. $\sigma_{\mathrm{R}} = \frac{S}{H}\sqrt{\frac{\sigma_S^2+\sigma_{B_S}^2/c^2}{(S-B_S/c)^2} + \frac{\sigma_H^2+\sigma_{B_H}^2/c^2}{(H-B_H/c)^2}}$,
  2. $\sigma_{\mathrm{C}} = \frac{1}{\ln(10)}\sqrt{\frac{\sigma_S^2+\sigma_{B_S}^2/c^2}{(S-B_S/c)^2} + \frac{\sigma_H^2+\sigma_{B_H}^2/c^2}{(H-B_H/c)^2}}$, and
  3. $\sigma_{\mathrm{HR}} = \dfrac{2\sqrt{(H-B_H/c)^2\left(\sigma_S^2+\sigma_{B_S}^2/c^2\right)+(S-B_S/c)^2\left(\sigma_H^2+\sigma_{B_H}^2/c^2\right)}}{\left[(H-B_H/c)+(S-B_S/c)\right]^2}$,

  where each σ is approximated, e.g., $\sigma_S \approx \sqrt{S+0.75}+1$.

Modeling the Hardness Ratios

- The typical Gaussian assumptions are inappropriate for few counts.
- Instead, we directly model photons from a source (η) and photons from background (β) as **independent Poisson variables**:

– $S = \eta_S + \beta_S \sim \text{Poisson}(\lambda_S+\xi_S)$, $H = \eta_H + \beta_H \sim \text{Poisson}(\lambda_H+\xi_H)$,

– $B_S \sim \text{Poisson}(c\,\xi_S)$, and $B_H \sim \text{Poisson}(c\,\xi_H)$.

where λ and ξ denote the expected source and background counts in the source region.

- Given the expected source counts, the hardness ratio is rewritten as:
  1. Simple counts ratio, $\mathrm{R} = \dfrac{\lambda_S}{\lambda_H}$,
  2. X-ray color, $\mathrm{C} = \log_{10}\dfrac{\lambda_S}{\lambda_H}$, and
  3. Fractional difference hardness ratio, $\mathrm{HR} = \dfrac{\lambda_H - \lambda_S}{\lambda_H + \lambda_S}$.

New Bayesian Method

Computing Posterior Distributions of Hardness Ratios:

- The posterior distributions of hardness ratios are computed from the joint posterior distribution of λ_S and λ_H:
  1. the posterior distribution of R is computed from

$$\begin{aligned} & p(\mathrm{R}, \lambda_H | S, H, B_S, B_H)\, d\mathrm{R}\, d\lambda_H \\ &= p(\lambda_S, \lambda_H | S, H, B_S, B_H) \left|\frac{\partial(\lambda_S, \lambda_H)}{\partial(\mathrm{R}, \lambda_H)}\right| d\lambda_S\, d\lambda_H \\ &= p(\mathrm{R}\lambda_H, \lambda_H | S, H, B_S, B_H)\lambda_H\, d\mathrm{R}\, d\lambda_H, \end{aligned}$$

  where we integrate out λ_H;

  2. the posterior distribution of C is computed from

$$\begin{aligned} & p(\mathrm{C}, \lambda_H | S, H, B_S, B_H)\, d\mathrm{C}\, d\lambda_H \\ &= p(\lambda_S, \lambda_H | S, H, B_S, B_H) \left|\frac{\partial(\lambda_S, \lambda_H)}{\partial(\mathrm{C}, \lambda_H)}\right| d\lambda_S\, d\lambda_H \\ &= p(10^{\mathrm{C}}\lambda_H, \lambda_H | S, H, B_S, B_H) 10^{\mathrm{C}} \ln(10) \lambda_H\, d\mathrm{C}\, d\lambda_H, \end{aligned}$$

  where we integrate out λ_H; and

  3. the posterior distribution of HR is computed from

$$\begin{aligned} & p(\mathrm{HR}, \omega | S, H, B_S, B_H)\, d\mathrm{HR}\, d\omega \\ &= p(\lambda_S, \lambda_H | S, H, B_S, B_H) \left|\frac{\partial(\lambda_S, \lambda_H)}{\partial(\mathrm{HR}, \omega)}\right| d\lambda_S\, d\lambda_H \\ &= p\left(\frac{(1-\mathrm{HR})\omega}{2}, \frac{(1+\mathrm{HR})\omega}{2} \middle| S, H, B_S, B_H\right) \frac{\omega}{2}\, d\mathrm{HR}\, d\omega, \end{aligned}$$

  where we integrate out $\omega = \lambda_S + \lambda_H$.

- To obtain the marginal posterior distribution of each hardness ratio, we use either **Monte Carlo integration** (Gibbs sampling) or **efficient numerical integration** (quadrature).

Simulation Study

Simulated Data Sets:

- To compare our Bayesian methods with the classical method, we suppose two imaginary sources with high and low counts:

High counts : $\lambda_S = \lambda_H = 30$, $\xi_S = \xi_H = 0.1$, and $c = 100$;

Low counts : $\lambda_S = \mu_H = 3$, $\xi_S = \xi_H = 0.1$, and $c = 100$.

- Based on the setting, we **ideally** have $S = \lambda_S$, $H = \lambda_H$, $B_S = c\,\xi_S$, and $B_H = c\,\xi_H$.

Simulation Results:

- Based on the ideal counts, we compute the point estimates with their associated error bars by using the classical and Bayesian methods.

	Method	Hardness ratio	True value	Point estimates: mode	Point estimates: mean	95% error bar: lower	95% error bar: upper
High counts	Classical method	R	1	1.0		0.3932	1.6068
		C	0	0.0		-0.2635	0.2635
		HR	0	0.0		-0.3034	0.3034
	Bayesian method	R	1	0.9376	1.0315	0.5622	1.5729
		C	0	0.0000	0.0000	-0.2590	0.2590
		HR	0	0.0000	0.0004	-0.2110	0.2123
Low counts	Classical method	R	1	1.0		**-1.8070**	3.8070
		C	0	0.0		-1.2191	1.2191
		HR	0	0.0		**-1.4035**	**1.4035**
	Bayesian method	R	1	0.5870	1.3192	0.0000	3.7081
		C	0	0.0000	0.00000	-0.6719	0.6719
		HR	0	-0.0003	-0.0005	-0.6666	0.6699

Table 1: Comparision Between Classical and Bayesian Methods.

- With large counts, the Bayesian method provides **more precise** error bars for the hardness ratios than the classical method, although both methods yield similar results.
- The classical method fails in **low-count cases** because the error bars are unreliable, as compared to the Bayesian methods.
- The posterior mode is more robust for R and C in terms of the so-called mean square error (MSE = variance + bias^2); the posterior mean is more robust for HR.
- As shown in Figure 1, the X-ray color exhibits the most **symmetric** posterior distribution.

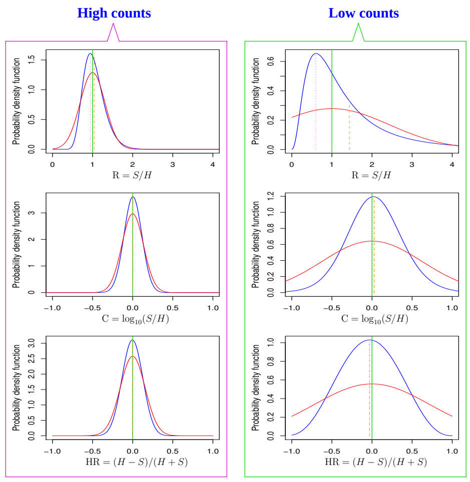

Figure 1: Distributions of Hardness Ratios for High and Low Counts. The solid **green** lines represent the true values of hardness ratios. The Bayesian method yields the posterior distribution for each hardness ratio (the solid **blue** line), the posterior mode (the dotted **pink** line), and the posterior mean (the dashed **orange** line). The classical method uses the Gaussian approximation to the sampling distribution for each hardness ratio (the **red** solid line).

Real Data Analysis

Globular Cluster Terzan-5

- One of the most important results from *Chandra* observations of globular clusters is the detection of quiescent Low-Mass X-ray binaries (qLMXBs; e.g., Heinke *et al.*, 2005, ApJ, 622, 556).
- These sources are characterized based on their soft spectrum (which, however, may be influenced by a hard power-law component).
- We applied our method on the point sources detected in a *Chandra* observation of the globular cluster Terzan-5 (PI J. Grindlay).
- We find 7 sources with luminosities above 10^{32} erg/s (0.5-6.0 keV), and X-ray colors between 0.5 and 2.5. (corresponding to neutron star atmosphere spectrum with 0% to 20% contribution by a power-law component with Gamma=1.5).
- Therefore these sources could be candidate qLMXBs.

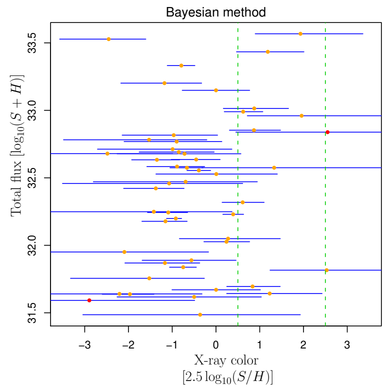

Figure 2: X-ray CMD Maded by the Bayesian Method. The classical method fails to compute the hardness ratios for two weak sources (marked by the solid **red** circles) because of non-detections in one of the two sub-bands.

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